Wave propagation in fractional nonlocal elastic continua

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Abstract: In this paper, the wave propagation in one-dimensional elastic continua, characterized by nonlocal interactions, is investigated by means of a fractional calculus approach. Derivatives of a non-integer order $1 < \alpha < 2$ with respect to the spatial variable are involved in the governing equation.

1. INTRODUCTION

The elastic behaviour of non-local continua has recently been investigated in a fractional calculus framework (Di Paola and Zingales, 2008; Carpinteri et al. 2009, 2011). This approach is original in many respects. With reference to classical nonlocal elastic approaches (Eringen and Edelen, 1972), the novelty is that the departure from local elasticity is obtained by lowering the order of (fractional) derivation of the displacement function in the governing equation (Lazopoulos, 2006; Tarasov and Zaslavsky, 2006; Atanackovic and Stankovic, 2009). On the other hand, compared with other fractional calculus applications in mechanics, the originality is that fractional derivatives are taken with respect to the spatial variable (and not with respect to the time variable, as it occurs, for instance, in visco-elasticity (Carpinteri and Mainardi, 1997; Mainardi, 2010)). Another important feature is that fractional operators have a clear mechanical meaning, i.e. they describe the interactions between non-adjacent points of the body by means of linear elastic springs whose stiffness decays as a power-law of the distance.

Limiting the analysis to a one-dimensional model, i.e. the fractional nonlocal elastic bar, the governing equation is a second order fractional differential equation in the displacement variable, where, beyond the usual second order derivative, a fractional derivative of order α with respect to the spatial variable appears. While in (Di Paola and Zingales, 2008) the range $0 < \alpha < 1$ was investigated, in (Carpinteri et al., 2011) the model was extended to the range $1 < \alpha < 2$. It was also highlighted that the range $1 < \alpha < 2$ provides a model within Eringen (integral) nonlocal elasticity framework (Eringen and Edelen, 1972), since it refers to a material whose stress is proportional to the fractional integral of the strain.

Starting from this result, in the present work the phenomenon of elastic wave propagation in a fractional nonlocal elastic bar is investigated for $1 < \alpha < 2$. The problem is analyzed for what concerns finite spatial domains: fractional finite differences (Yang et al., 2010) are involved in the discretization process. Eventually, the longitudinal resonant frequencies and the standing waves related to such nonlocal continua are evaluated. Solutions are examined and compared with the classical one, which is recovered by the present model as the order of fractional derivation coincides with the integer value ($\alpha = 2$).

2. ERINGEN NONLOCAL MODEL

According to Eringen nonlocal elasticity (Eringen and Edelen, 1972), the stress $\sigma(x)$ at a given point depends on the strain $\varepsilon(x)$ in a neighbourhood of that point by means of a convolution integral. This dependence is described by a proper attenuation function g, which decays along with the distance. In the case of a one-dimensional domain (i.e. a bar), the constitutive law reads:

$$\sigma(x) = E\left[\beta_1\varepsilon(x) + \beta_2\kappa_\alpha\int_a^b \varepsilon(y)g(x-y)\mathrm{d}y\right], \tag{1}$$

where $x = a$ and $x = b$ are the bar extreme coordinates, E the Young's modulus, ε the strain defined as the derivative of the longitudinal displacement u and κ_α is a material constant. The bar length is L ($L = b-a$). The parameters β_1 and β_2, as in the classical nonlocal approach (Polizzotto, 2001), weigh the local and the nonlocal contributions: $\beta_1 + \beta_2 = 1$, $0 \le \beta_1, \beta_2 \le 1$.

If the attenuation function g is taken in the form (Carpinteri et al., 2009a)

$$g(\xi) = \frac{1}{2\Gamma(2-\alpha)\,|\,\xi\,|^{\alpha-1}}, \tag{2}$$

with $1 < \alpha < 2$, the constitutive relationship becomes:

$$\sigma = E\left[\beta_1\varepsilon + \beta_2\kappa_\alpha\left(I_{a,b}^{2-\alpha}\,\varepsilon\right)\right], \tag{3}$$

where the operator $I_{a,b}^{\beta}$ is the fractional Riesz integral ($\beta > 0$, (Samko et al., 1993)) :

$$I_{a,b}^{\beta} f(x) = \frac{1}{2\Gamma(\beta)}\int_a^b \frac{f(y)}{\left|x-y\right|^{1-\beta}}\,\mathrm{d}y\,. \tag{4}$$

The constant κ_α has hence anomalous physical dimensions $[\mathbf{L}]^{\alpha-2}$ and the following condition holds, for the sake of completeness: $\kappa_\alpha = 1$ for $\alpha = 2$.

Equation (3) reverts to the classical constitutive relationship for $\alpha = 2$,

$$\sigma = \left(\beta_1 + \beta_2\right) E\varepsilon = E\varepsilon, \tag{5}$$

while, for $\alpha = 1$, it provides

$$\sigma = \beta_1 E\varepsilon + \frac{\beta_2 E \kappa_\alpha}{2}(u_b - u_a), \tag{6}$$

which describes the behavior of a bar possessing a reduced Young's modulus $\beta_1 E$ with a spring of stiffness $\beta_2 EA\ \kappa_\alpha/2$ connecting its extremes, A being the bar cross-section.

In order to get the equilibrium equation in terms of the displacement function $u(x)$, we simply need to substitute Eq. (3) into the static equation $\mathrm{d}\sigma/\mathrm{d}x + f(x) = 0$, where $f(x)$ is the longitudinal force per unit volume. By exploiting the definitions of the Riemann-Liouville fractional derivatives ($1 < \beta < 2$, (Podlubny, 1999))

$$D_{a+}^{\beta} f(x) = \frac{f(a)}{\Gamma(1-\beta)(x-a)^{\beta}} + \frac{f'(a)}{\Gamma(2-\beta)(x-a)^{\beta-1}} + \frac{1}{\Gamma(2-\beta)}\int_a^x \frac{f''(y)}{(x-y)^{\beta-1}}\,\mathrm{d}y, \tag{7}$$

$$D_{b-}^{\beta} f(x) = \frac{f(\mathrm{b})}{\Gamma(1-\beta)(\mathrm{b}-x)^{\beta}} - \frac{f'(\mathrm{b})}{\Gamma(2-\beta)(\mathrm{b}-x)^{\beta-1}} + \frac{1}{\Gamma(2-\beta)}\int_x^b \frac{f''(y)}{(\mathrm{y}-x)^{\beta-1}}\,\mathrm{d}y, \tag{8}$$

some more analytical manipulations lead to (Atanackovic and Stankovic, 2009; Carpinteri et al., 2011):

$$\beta_1 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \beta_2 \frac{\kappa_\alpha}{2}\left\{D_{a+}^{\alpha}[u(x)-u(a)] + D_{b-}^{\alpha}[u(x)-u(b)]\right\} = -\frac{f(x)}{E}. \tag{9}$$

The term in the curly brackets is equal to $2u''$ when $\alpha = 2$, and vanishes when $\alpha = 1$. Equation (9) is a fractional differential equation, whose solution, obtained by means of fractional finite differences, was provided in (Carpinteri et al., 2011).

Notice that Eq.(9) can be expressed through the Caputo fractional derivatives (Carpinteri and Mainardi, 1997) and that it can be rewritten, by exploiting the definition of the Marchaud fractional derivatives (Samko et al., 1993), as:

$$\beta_1 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \beta_2 \frac{\kappa_\alpha(\alpha-1)}{2\,\Gamma(2-\alpha)} \times \left[\frac{u(x)-u(a)}{(x-a)^{\alpha}} + \frac{u(x)-u(b)}{(b-x)^{\alpha}} + \alpha\int_a^b \frac{u(x)-u(\mathrm{y})}{|x-y|^{1+\alpha}}\,\mathrm{d}y\right] = -\frac{f(x)}{E}. \tag{10}$$

An analogous equation to Eq. (10), although retaining only the integral term in the square brackets, was considered in (Di Paola and Zingales, 2008; Cottone et al., 2009) for $0 < \alpha < 1$. However, according to the present framework, the physical validity of the model is questionable, since the function $g(\xi)$ (Eq. (2)) is no more an attenuation function.

2.1 Equivalent point-spring model

On the basis of the analysis presented in (Di Paola and Zingales, 2008), a physical interpretation of the governing equation (10) was given in (Carpinteri et al., 2011). Let us introduce a partition of the interval $[a, b]$ on the x axis made of n ($n \in N$) intervals of length $\Delta x = L/n$. The generic point of the partition has the abscissa x_i, with $i = 1, \dots, n+1$ and $x_1 = a$, $x_{n+1} = b$; that is, $x_i = a + (i-1)\Delta x$. Hence, for the inner points of the domain ($i = 2, \dots, n$), the discrete form of Eq. (10) reads:

$$k_{i,i+1}^{l}(u_i - u_{i+1}) + k_{i,i-1}^{l}(u_i - u_{i-1}) + k_{i,1}^{vs}(u_i - u_1) + k_{i,n+1}^{vs}(u_i - u_{n+1}) + \sum_{j=1, j\neq i}^{n+1} k_{i,j}^{vv}(u_i - u_j) = f_i A \Delta x, \tag{11}$$

where $u_i \equiv u(x_i)$ and $f_i \equiv f(x_i)$. It is evident how the nonlocal fractional model is equivalent to a point-spring model where three kinds of springs appear: the local springs, ruling the local interactions, whose stiffness is k^l; the springs connecting the inner material points with the bar edges, ruling the volume-surface long-range interactions, with stiffness k^{vs}; the springs connecting the inner material points each other, describing the nonlocal interactions between non-adjacent volumes, whose stiffness is k^{vv}. Provided that the indexes are never equal one to the other, the following expressions for the stiffnesses hold ($i = 1, \dots, n+1$):

$$k_{i,i+1}^{l} = k_{i+1,i}^{l} = \beta_1 EA/\Delta x, \tag{12}$$

$$k_{i,1}^{vs} = k_{1,i}^{vs} = \beta_2 EA\kappa_\alpha \frac{\alpha-1}{2\,\Gamma(2-\alpha)} \frac{\Delta x}{(x_i - x_1)^{\alpha}}, \tag{13}$$

$$k_{i,n+1}^{vs} = k_{n+1,i}^{vs} = \beta_2 EA\kappa_\alpha \frac{\alpha-1}{2\,\Gamma(2-\alpha)} \frac{\Delta x}{(x_{n+1} - x_i)^{\alpha}}, \tag{14}$$

$$k_{i,j}^{vv} = k_{j,i}^{vv} = \beta_2 EA\kappa_\alpha \frac{\alpha(\alpha-1)}{2\,\Gamma(2-\alpha)} \frac{(\Delta x)^2}{|x_i - x_j|^{1+\alpha}}. \tag{15}$$

Furthermore, by exploiting the Principle of Virtual Work to derive the proper either kinematic or static boundary conditions, it is possible to show that a fourth set of springs has to be introduced. It is composed by a unique spring connecting the two bar extremes with the stiffness:

$$k_{1,n+1}^{ss} = k_{n+1,1}^{ss} = \frac{\beta_2 EA\kappa_\alpha}{2\,\Gamma(2-\alpha)} \frac{1}{(x_{n+1} - x_1)^{\alpha-1}}. \tag{16}$$

The superscript ss for the stiffness (16) is used since the spring connecting the bar edges can be seen as modelling the interactions between material points lying on the surface, which, in the simple one-dimensional model under examination, reduce to the two points $x = a, b$. Note that the presence of such a spring was implicitly embedded in the constitutive equation (3). However, since it provides a constant stress contribution throughout the bar length, its presence was lost by derivation when inserting the constitutive relationship into the differential equilibrium equation, i.e. when passing from Eq. (3) to Eq. (9).

To summarize, the constitutive fractional relationship (3) is equivalent to a point-spring model with four sets of springs, one local (12) and three nonlocal (13)-(16). Note that their stiffnesses all decay with the distance, although the decaying velocity differs from one kind to the other. The equivalent point-spring model is drawn in Fig.1 for $n = 4$.

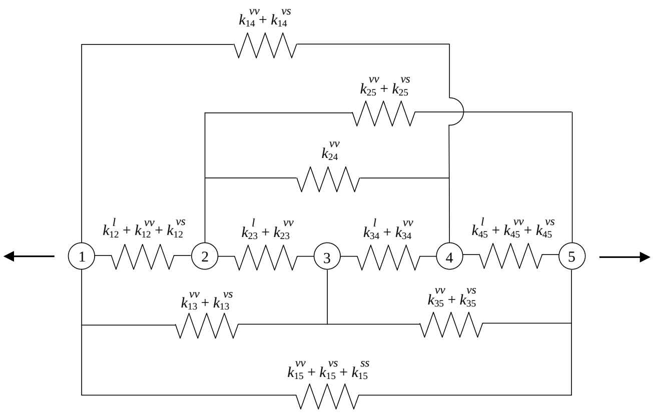

Fig. 1. Point-spring model equivalent to the nonlocal fractional elastic bar ($n = 4$).

For what concerns the limit cases, $\alpha = 2$ corresponds to the classical local elastic bar. In fact, if $\alpha \to 2^-$, since $\Gamma(0^+) = +\infty$, the surface-surface (Eq. (16)) and the volume-surface (Eqs. (13)-(14)) contributions disappear. For what concerns the interactions between inner material points (Eq. (15)), only the interactions between adjacent material points are retained (the Gamma function tends to infinity, but the integral in Eq. (10) diverges). Correspondingly, the additive term in Eq. (3) has the same form as the classical (local) one, the model representing a bar with a Young's modulus equal to $E(\beta_1 + \beta_2) = E$, while the governing equation (10) becomes $u'' = f/E$.

On the other hand, if $\alpha \to 1^+$, the volume-volume and the volume-surface spring interactions ruled by Eqs. (13)-(15) vanish, and only the contribution (16) remains (together with the local springs (12)): the nonlocal model corresponds to a classical elastic bar with a reduced Young's modulus $\beta_1 E$ in parallel with a spring of stiffness $\beta_2 E A \kappa_\alpha /2$. The governing equation reverts to: $u'' = f/(\beta_1 E)$.

3. WAVE PROPAGATION

By means of simple equilibrium considerations, or through a variational approach similar to that performed in (Cottone et al., 2009), the fractional wave equation on a bar of finite length takes the following expression:

$$u_{tt}(x,t) = c^2 (\beta_1 u_{xx}(x,t) + \\ + \beta_2 \frac{\kappa_\alpha}{2} \{ {}_x D_{a+}^{\alpha}[u(x,t) - u(a,t)] + {}_x D_{b-}^{\alpha}[u(x,t) - u(b,t)] \}), \tag{17}$$

where t is the time variable, $c = \sqrt{E/\rho}$ is the well-known propagation speed of the wave (ρ being the bar volumetric density), and ${}_x D_{a+}^{\alpha}$ and ${}_x D_{b-}^{\alpha}$ are the fractional derivatives with respect to the spatial variable x. The conventions $[\cdot]_{tt} = \partial^2 / \partial t^2$ and $[\cdot]_{xx} = \partial^2 / \partial x^2$ are now adopted, for the sake of simplicity. No external forces are considered in Eq. (17).

For $\alpha = 2$, as already discussed, the constitutive relationship is the classical one and Eq. (17) becomes:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t). \tag{18}$$

On the other hand, for $\alpha = 1$ the term in the curly brackets vanishes and Eq. (17) provides

$$u_{tt}(x,t) = c^2 \beta_1 u_{xx}(x,t) = \frac{\beta_1 E}{\rho} u_{xx}(x,t) \tag{19}$$

i.e., the wave equation on a local bar with a reduced Young's modulus. Consequently, also the wave propagation speed results reduced. Suitable initial and boundary conditions must be assigned to Eq. (17) (Carpinteri et al., 2009a).

Analytical solutions of the fractional wave equation can be obtained through the Laplace-Fourier transforms (Atanackovic and Stankovic, 2009), for what concerns infinite space domains.

On the other hand, since a bar of finite length is taken into account in the present case, the problem is faced by numerical schemes. Since the order of derivation is comprised between 1 and 2, we chose to implement the so-called L2 algorithm firstly proposed by Oldham and Spanier (1974) and later applied to fractional diffusion equations by Yang et al. (2010). The L2 algorithm is based on the formulae (7) and (8). By approximating the first and the second order derivatives by means of the usual finite differences and evaluating analytically the remaining part of the integrals in Eqs. (7-8), we get the following approximate discrete expressions of the fractional derivatives in the internal points of the domain $[a, b]$, i.e. for $i = 2,\ldots,n$ ($1 \le \alpha < 2$) :

$$ {}_xD^{\alpha}_{a+} f(x_i,t_j) \approx \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)} \times \\ \times \left\{ \frac{(1-\alpha)(2-\alpha)}{(i-1)^{\alpha}} f_{1,j} + \frac{2-\alpha}{(i-1)^{\alpha-1}} (f_{2,j} - f_{1,j}) + \\ + \sum_{k=0}^{i-2} (f_{i-k+1,j} - 2 f_{i-k,j} + f_{i-k-1,j})[(k+1)^{2-\alpha} - k^{2-\alpha}] \right\} \tag{20} $$

$$ {}_xD^{\alpha}_{b-} f(x_i,t_j) \approx \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)} \times \\ \times \left\{ \frac{(1-\alpha)(2-\alpha)}{(n-i+1)^{\alpha}} f_{n+1,j} - \frac{2-\alpha}{(n-i+1)^{\alpha-1}} (f_{n+1,j} - f_{n,j}) + \\ + \sum_{k=0}^{n-i} (f_{i+k+1,j} - 2 f_{i+k,j} + f_{i+k-1,j})[(k+1)^{2-\alpha} - k^{2-\alpha}] \right\}. \tag{21} $$

Notice that t_j is equal to $j\Delta t$, with $j = 1,\ldots,m + 1$, where $\Delta t = T/m$ represents the discretization step of the time domain $[0, T]$.

The final algorithm to solve Eq. (17) can be written starting from that proposed for the classical wave equation ($\alpha = 2$), by properly taking into account the contributions provided by Eqs. (20-21).

We have applied the developed fractional nonlocal model to investigate the wave propagation in a clamped bar of length L ($a = 0$, $b = L$) subjected to a prescribed sinusoidal displacement at the right extreme ($b = L$). The following conditions are assigned to Eq.(17):

$$ \gamma_0(x) = \gamma_1(x) = 0, \qquad u_a(t) = 0, \qquad u_b(t) = U \sin \omega t. \tag{22} $$

where U and ω are the amplitude and the frequency of the forcing term, respectively.

The parameters used for computations are: L = 5m, A = 0.1m^2, M = 5Kg, (thus, $\rho = M/(AL)$ = 10Kg/m^3), E = 10N/m^2, $\beta_1 = 0.1$, κ_α = 1 m$^{\alpha-2}$, T = 5s, U = 0.001m and $\omega = \pi/50$ Hz. Moreover, n and m are chosen equal to 200 and 300, respectively: thus $\Delta t / \Delta x < 1$, and the numerical scheme stability is guaranteed.

Results are presented in Fig. 2 and 3 for what concerns $\alpha = 2$, and 1.5, respectively. For non-integer orders of derivation (Fig. 3), differently from the classical case, the wave shape changes during propagation. This is imputable to the presence of long-range interactions (Eqs. (13)-(16)), which prevent the formation of a marked wavefront.

3.1 Resonant frequencies and standing waves

Let us consider again the wave propagation in a clamped bar subjected to a prescribed sinusoidal displacement at the free-end. As the wave approaches the fixed end, it starts to reflect back in the opposite direction along the bar and to interfere with the incident wave. If the forcing frequency ω coincides with one of the resonant frequencies ω_r of the structure, the asymptotic condition is a standing elastic wave, i.e., a wave that remains in a constant configuration. In other words, the steady-state motion takes the following form:

$$ u(x,t) \propto \psi_r(x) \sin \omega_r t \,, \tag{23} $$

where $\psi_r(x)$ represents the r-standing wave pattern i.e., the r-natural mode of the structure. In order to evaluate both ω_r and $\psi_r(x)$, we must solve the well-known eigenvalue problem

$$ (\mathbf{K} - \omega_r^2 \mathbf{M})\psi_r = 0, \tag{24} $$

K and **M** being the stiffness and mass matrices, respectively. Equation (24) possesses a non trivial solution if and only if the following condition is satisfied:

$$ \mathrm{Det}(\mathbf{K} - \omega_r^2 \mathbf{M}) = 0. \tag{25} $$

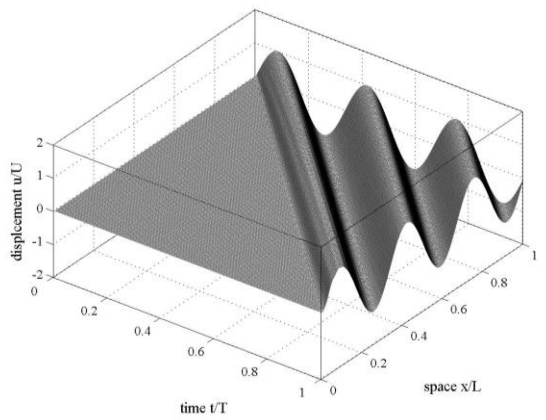

Fig. 2. Wave propagation in a clamped local bar subjected to a prescribed sinusoidal displacement at the free-end: $\alpha = 2$.

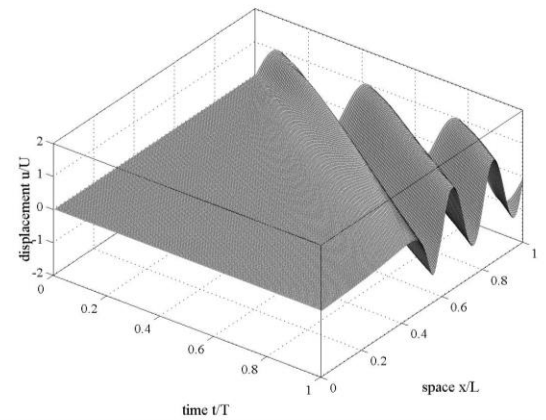

Fig. 3. Wave propagation in a clamped local bar subjected to a prescribed sinusoidal displacement at the free-end: $\alpha = 1.5$.

Once the eigenvalues ω_r^2 are obtained through Eq. (25), they can be inserted into Eq. (24) to get the corresponding eigenvectors ψ_r, i.e., the standing waves.

The analysis presented above is now applied to fractional nonlocal continua. Notice that, equivalently, the problem can be faced by solving the fractional differential equation

$$\omega_r^2 \psi_r(x) + c^2 \left(\beta_1 \frac{\mathrm{d}^2 \psi_r}{\mathrm{d}x^2} + \right.$$
$$\left. + \beta_2 \frac{\kappa_\alpha}{2} \left\{ D_{a+}^{\alpha} [\psi_r(x) - \psi_r(a)] + D_{b-}^{\alpha} [\psi_r(x) - \psi_r(b)] \right\} = 0 \right), \quad (26)$$

obtained by substituting Eq. (23) into Eq. (17), with appropriate boundary conditions.

The stiffness matrix $\mathbf{K}$ of a nonlocal bar, according to what has been shown in Section 3, is the sum of four stiffness matrices:

$$\mathbf{K} = \mathbf{K}^{l} + \mathbf{K}^{vv} + \mathbf{K}^{vs} + \mathbf{K}^{ss} \quad (27)$$

whose non-diagonal terms are provided by the opposite of the corresponding stiffnesses (12-16). Furthermore, the diagonal terms $k_{i,i}$ of each matrix are given by the relationship:

$$k_{i,i} = \sum_{j=1,\, j \neq i}^{n+1} k_{i,j}, \qquad i = 1, \ldots, n+1. \quad (28)$$

All the four matrices at the right-hand side of Eq. (27) are symmetrical, with positive elements on the diagonal and negative outside. More in detail, the local matrix $\mathbf{K}^{l}$ is tridiagonal; the nonlocal matrix $\mathbf{K}^{vv}$ ruling the long-range interactions between inner points is fully populated; the nonlocal matrix related to the inner-outer interactions $\mathbf{K}^{vs}$ has only border and diagonal elements different from zero; finally, the nonlocal matrix $\mathbf{K}^{ss}$ describing the interaction between the bar edges is empty except for the four corner elements. On the other hand, $\mathbf{M}$ is assumed to be a diagonal matrix, whose elements are all equal to $M/(n+1)$.

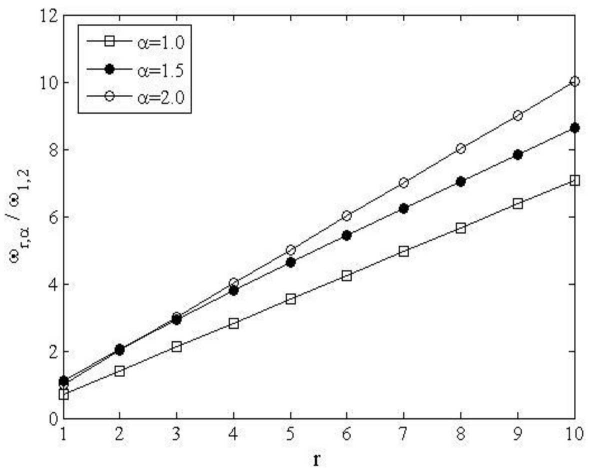

Fig. 4. First ten resonant frequencies of a double clamped nonlocal bar, for different fractional orders *α*.

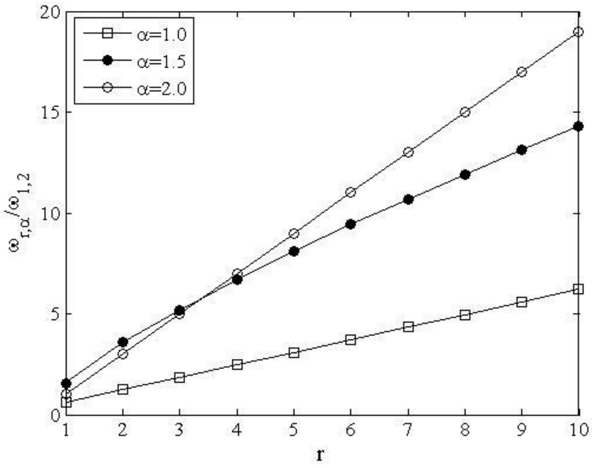

Fig. 5. First ten resonant frequencies of a single clamped nonlocal bar, for different fractional orders *α*.

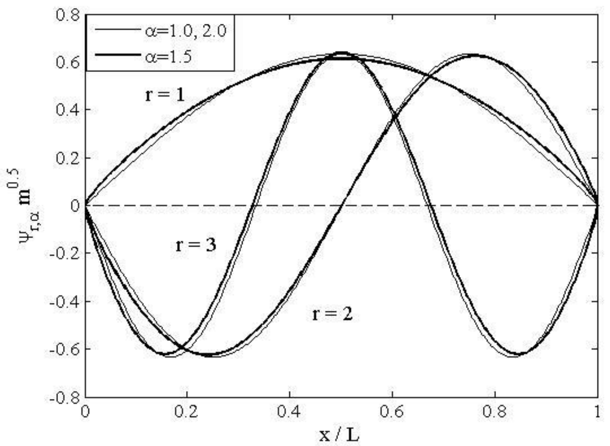

Fig. 6. First three standing waves of a double clamped nonlocal bar, for different fractional orders *α*.

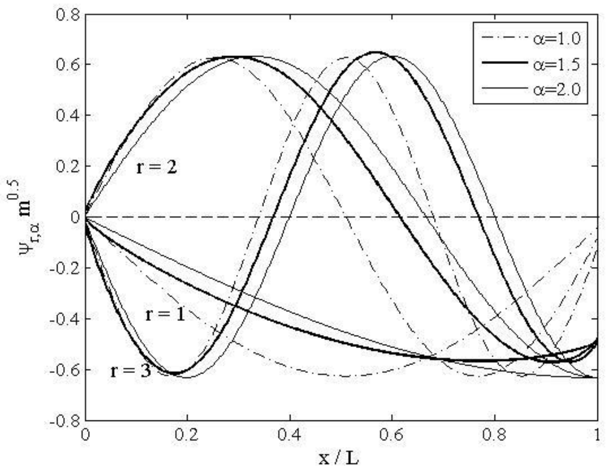

Fig. 7. First three standing waves of a single clamped nonlocal bar, for different fractional orders *α*.

Let us start by considering a double clamped bar. The constraint conditions can be expressed by deleting the first and the last rows and columns in the matrices **K** and **M**. Thus, in the present case, $\mathbf{K}^{ss}$ provides no contributions. The first ten natural frequencies obtained by solving Eq. (25), for different fractional orders, are plotted in Fig. 4. If $\alpha = 2$, as in the classical case (Eq. (5)), it is found that:

$$\omega_{r,\alpha=2} = r\frac{\pi}{L}\sqrt{\frac{E}{\rho}}, \qquad r = 1,2,3,... \tag{29}$$

On the other hand, for α = 1, we get :

$$\omega_{r,\alpha=1} = r\frac{\pi}{L}\sqrt{\frac{\beta_1 E}{\rho}}, \qquad r = 1,2,3,... \tag{30}$$

since the second contribution in Eq. (6) vanishes. For non-integer values of α, the relation $\omega_{r,\alpha}$ vs. r is not linear: the resonant frequencies are comprised, at least for the higher modes, between those evaluated through Eqs. (29) and (30).

An analogous situation is recovered if a single-clamped bar is considered. Results are presented in Fig. 5. While Eq. (29) still holds, provided that r is substituted by $r - 0.5$, a generalization of Eq. (30) is not so straightforward, since the matrix $\mathbf{K}^{ss}$ contribution is different from zero (see Eq. (6))

Eventually, the orthonormal standing waves ψ_r, with r = 1, 2 and 3, are evaluated through Eq. (24) for both a double and a single clamped bar (Figs. 6 and 7, respectively). In the former case, the natural modes related to α = 1 and 2 coincide (Fig. 6), since they do not depend on the Young's modulus. In the latter case, due to the interaction between the bar edges, the standing waves depart more and more from the local ones (α = 2) as α decreases from 2 to 1 (Fig. 7).

6. CONCLUSIONS

In the present paper, the wave propagation in nonlocal continua has been investigated by means of a fractional approach. Starting from the constitutive relationship, expressing the nonlocal dependency of stress on strain, by means of simple equilibrium considerations, the fractional differential wave equation has been derived. The problem has been faced numerically, by means of fractional finite differences. It has been shown that the presence of nonlocal fractional interactions affects the wave propagation in one-dimensional elastic continua: the wave shape is deformed, the resonant frequencies show a non-linear behaviour as the number of modes increases and the standing waves deviate from the classical local ones. This effect is more pronounced if at least one of the bar edges is not restrained, since the contribution of nonlocal interactions between outer and inner points results more significant.

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